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Properties of the canonical transformations of the time for the Toda lattice and the Henon–Heiles system

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Abstract. For the Toda lattice and the Henon–Heiles system we consider properties of the canonical transformations of the extended phase space, which preserve integrability. At the special values of integrals of motion the integral trajectories, separated variables and the action variables are invariant under change of the time. On the other hand, mapping of the time induces a shift of the generating function of the Bäcklund transformation.

1. Introduction

Let us consider some integrable system on a $2n$-dimensional symplectic manifold $\mathcal{M}$ endowed with a symplectic form Ω and some coordinates $\{p_j, q_j\}_{j=1}^n$. Introduce transformations of the time and the Hamilton function

$$\begin{aligned} &t \mapsto \widetilde{t} \qquad && \mathrm{d}\widetilde{t} = v(p,q)\,\mathrm{d}t \\ &H \mapsto \widetilde{H} \qquad && \widetilde{H} = v(p,q)^{-1} H \end{aligned} \tag{1.1}$$

which map a given completely integrable system into the other completely integrable system on $\mathcal{M}$. For instance, the Maupertuis–Jacobi mapping [2] and the Kepler change of the time [19, 20] belong to such transformations.

The transformations (1.1) change the initial equations of motion

$$\frac{\mathrm{d}q_i}{\mathrm{d}\widetilde{t}} = v^{-1}(p,q)\left(\frac{\mathrm{d}q_i}{\mathrm{d}t} - \widetilde{H}\frac{\partial v}{\partial p_i}\right) \qquad \frac{\mathrm{d}p_i}{\mathrm{d}\widetilde{t}} = v^{-1}(p,q)\left(\frac{\mathrm{d}p_i}{\mathrm{d}t} + \widetilde{H}\frac{\partial v}{\partial q_i}\right)$$

but preserve the canonical form of the Hamilton–Jacobi equation

$$\frac{\partial \mathcal{S}}{\partial t} + H = 0 \qquad \text{where} \quad \mathcal{S} = \int \big(p\dot{q} - H(p,q)\big)\,\mathrm{d}t. \tag{1.2}$$

Since, transformations (1.1) may be called canonical transformations of the extended phase space $\mathcal{M}_E$ [19]. However, as the Maupertuis–Jacobi [2] and the Kepler transformations [19], these transformations do not in general retain the corresponding differential 2-form $\Omega = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i - \mathrm{d}H \wedge \mathrm{d}t$ in the extended phase space $\mathcal{M}_E$.

On a $2n$-dimensional symplectic manifold $\mathcal{M}$ solution of the Hamilton–Jacobi equation (1.2) is an n-dimensional Lagrangian submanifold $\mathcal{C}^{(n)}$ [21] lying on the fixed energy surface

$$\mathcal{C}^{(n)} \subset \mathcal{Q}^{2n-1} : \ \big(H(p,q) = E\big).$$

Recall that a Lagrangian submanifold is one where the symplectic form Ω vanishes when restricted to it $\Omega|_{\mathcal{C}} = 0$. This definition is completely invariant with respect to change of local coordinates and the implicit representation of the Lagrangian submanifold [1, 21]. The time t is an external parameter related to some parametrization of the Lagrangian foliation.

Of course, starting from solution $\mathcal{C}^{(n)}$ we cannot reconstruct the whole fixed energy surface $\mathcal{Q}^{2n-1}$ and the Hamilton–Jacobi equation (1.2). So, due to the Maupertuis principle, the time t and the corresponding Hamilton function H cannot be restored from a given Lagrangian foliation $\mathcal{C}^{(n)}$ [1]. In this and several other unexpected situations in mathematics, dynamics is occasionally invading mathematical objects in which time is not present in the definition and yet the object can be endowed with various dynamical system structures, which are continuous or discrete.

Canonical transformations of the extended phase space (1.1) determine various parametrizations of a given Lagrangian submanifold. Each new parametric form of $\mathcal{C}^{(n)}$ yields a new Hamiltonian system related to the same geometric object. So, we can suppose that these different integrable systems have some common properties.

Usually, the Lagrangian submanifold depends on the $n+m$ arbitrary constants. The n constants $\alpha_1, \dots, \alpha_n$ are identified with the values of integrals of motion $I_j = \alpha_j$ [21], while the remaining m constants $a_1, \dots, a_m$ are free parameters. Below we will discuss a special class of different parametric forms of a Lagrangian submanifold, which is associated with mutual permutations of energy E and parameter a_k.

The passage from a given parametrization to the another one gives rise to the transformations of all the properties of integrable systems, such as integrals of motion, Lax equations and r-matrix algebras, separated variables and the action-angles variables [17–20].

In this paper we continue to study these induced transformations. The aim of this paper is to understand some general features of the canonical transformations (1.1) of the extended phase space for such different integrable systems as the Toda lattice and the Henon–Heiles system, which belong to the family of integrable Stäckel systems.

2. The Henon–Heiles integrable systems

The equations of motion of the Henon–Heiles system [14] derive from the following Hamilton function:

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + 2ax(\beta x^2 + 3y^2 + b) \qquad a, b \in \mathbb{R} \tag{2.1}$$

using the standard Hamilton equations

$$\dot{z} = \frac{\partial H}{\partial p_z} = p_z \qquad \dot{p}_z = -\frac{\partial H}{\partial z} \qquad z = x, y. \tag{2.2}$$

Only three integrable cases are known [14]

$$\text{(a) } \beta = 1 \qquad \text{(b) } \beta = 6 \qquad \text{(c) } \beta = 16 \tag{2.3}$$

while the remaining parameters a and b be an arbitrary constants.

According to [14, 17], canonical transformation of the extended phase space (1.1)

$$v = x \qquad \mathrm{d}\widetilde{t} = x\,\mathrm{d}t \qquad \widetilde{H} = x^{-1}(H - c) \qquad c \in \mathbb{R} \tag{2.4}$$

preserves integrability and maps the Henon–Heiles system into the other integrable system. Mapping (2.4) induces the following almost multiplicative transformations of the initial equations of motion (2.2):

$$\frac{\mathrm{d}x}{\mathrm{d}\widetilde{t}} = v^{-1}\dot{x} \qquad \frac{\mathrm{d}p_x}{\mathrm{d}\widetilde{t}} = v^{-1}(\dot{p}_x + \widetilde{H}) \qquad \frac{\mathrm{d}y}{\mathrm{d}\widetilde{t}} = v^{-1}\dot{y} \qquad \frac{\mathrm{d}p_y}{\mathrm{d}\widetilde{t}} = v^{-1}\dot{p}_y.$$

After a canonical change of variables

$$x = \left(\tfrac{3}{2}\boldsymbol{x}\right)^{2/3} \quad p_x = \boldsymbol{p}_x \left(\tfrac{3}{2}\boldsymbol{x}\right)^{1/3} \qquad y = -2\sqrt{3a}\boldsymbol{y} \quad p_y = \frac{\boldsymbol{p}_y}{2\sqrt{3a}}$$

and a rescaling of parameters

$$a \to \tfrac{1}{4}a\left(\tfrac{2}{3}\right)^{1/3} \qquad b \to \frac{b}{2a} \qquad c \to ac\left(\tfrac{2}{3}\right)^{-2/3}$$

the new Hamilton function $\widetilde{H}$ (2.4) becomes

$$\widetilde{H} = \tfrac{1}{2}(\boldsymbol{p}_x^2 + \boldsymbol{p}_y^2) + a\boldsymbol{x}^{-2/3}\left(\tfrac{3}{4}\beta\boldsymbol{x}^2 + \boldsymbol{y}^2 - c\right) + b. \tag{2.5}$$

This Hamilton function describes the so-called Drach–Holt system [14, 17]. At $\beta = 1$ and at $\beta = 16$ the corresponding Lax matrices are 3×3 matrices and the spectral curves are trigonal algebraic curves [6]. We shall consider these more complicated cases in the forthcoming publications.

According to [6, 17], at $\beta = 6$ the 2×2 Lax matrix $\mathcal{L}(\lambda)$ for the Henon–Heiles system is equal to

$$\mathcal{L}(\lambda) = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & -\mathbb{A} \end{pmatrix}(\lambda) = \begin{pmatrix} \dfrac{p_x}{2} - \dfrac{p_y y}{4\lambda} & \lambda + x - \dfrac{y^2}{4}\lambda \\ \dfrac{p_y^2}{4}\lambda & -\dfrac{p_x}{2} + \dfrac{p_y y}{4\lambda} \end{pmatrix}$$
$$+6a \begin{pmatrix} 0 & 0 \\ \lambda^2 - x\lambda + x^2 + \tfrac{1}{4}y^2 + \tfrac{1}{6}b & 0 \end{pmatrix} \tag{2.6}$$

whereas the second matrix in the Lax equation $\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{A}]$ is given by

$$\mathcal{A}(\lambda) = \begin{pmatrix} 0 & 1 \\ 6a(\lambda - 2x) & 0 \end{pmatrix}.$$

Canonical transformation of the extended phase space (2.4) gives rise to the mapping of these Lax matrices

$$\widetilde{\mathcal{L}}(\lambda) = \mathcal{L}(\lambda) - \frac{1}{2}\widetilde{H}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \widetilde{\mathcal{A}}(\lambda) = \frac{1}{x}\mathcal{A}(\lambda) \tag{2.7}$$

and the following transformations of the corresponding hyperelliptic spectral curves:

$$\begin{aligned} \mathcal{C}\colon \mu^2 &= P(\lambda) = 6a\lambda^3 + ab\lambda + \frac{H}{2} + \frac{K}{\lambda} \\ \widetilde{\mathcal{C}}\colon \mu^2 &= \widetilde{P}(\lambda) = 6a\lambda^3 + \left(ab - \tfrac{1}{2}\widetilde{H}\right)\lambda + \frac{c}{2} + \frac{\widetilde{K}}{\lambda}. \end{aligned} \tag{2.8}$$

The corresponding transformation of the r-matrix Poisson brackets has been considered in [17].

Substituting some fixed values of integrals of motion into the symmetric product of spectral curves (2.8) one obtains a Lagrangian submanifold $\mathcal{C}^{(2)} = \mathcal{C} \times \mathcal{C}$, which depends on parameters $a, b, \alpha_1 = H, \alpha_2 = K$ or $a, c, \beta_1 = \widetilde{H}, \beta_2 = \widetilde{K}$, respectively. Canonical transformation of the extended phase space (2.4) changes the parametric forms of the common Lagrangian submanifold only.

The separation variables $\{\lambda_1\lambda_2\}$ for both systems are zeros of the common non-diagonal entry of the Lax matrices $\mathcal{L}(\lambda)$ and $\widetilde{\mathcal{L}}(\lambda)$

$$\mathbb{B}(\lambda) = \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)}{\lambda} \tag{2.9}$$

and values of the second common entry

$$\mu_i = \mathbb{A}(\lambda_i) \qquad i = 1, 2$$

see references in [17].

The variables $\{\lambda_{1,2}, \mu_{1,2}\}$ are the standard parabolic coordinates, which lie on the hyperelliptic curves (2.8). Applying Arnold's method [1], action variables have the form

$$s_i = \oint_{A_i} \sqrt{P(\lambda)}\,\mathrm{d}\lambda \qquad \widetilde{s}_i = \oint_{\widetilde{A}_i} \sqrt{\widetilde{P}(\lambda)}\,\mathrm{d}\lambda \tag{2.10}$$

where A_i and $\widetilde{A}_i$ are A-cycles of the Jacobi variety of the algebraic curves (2.8), respectively. Thus, the Abel transformation linearizes equations of motion using first kind Abelian differentials on the corresponding hyperelliptic spectral curves.

Let the parameter b determine the potential of the Henon–Heiles system (2.1) and parameters $\widetilde{b}$ and c define the potential of the Holt system (2.5). At the special choice of values of integrals of motion

$$H = c \qquad \widetilde{H} = 2(\widetilde{b} - b) \qquad K = \widetilde{K} = \alpha$$

the initial spectral curve is equal to the resulting curve (2.8). Thus, as for the Maupertuis–Jacobi mapping [2], integral trajectories of the Henon–Heiles system coincide with the trajectories of the Holt system on the intersection of the corresponding common levels of integrals $\mathcal{M}_\alpha$ and $\widetilde{\mathcal{M}}_\alpha$. In the neighbourhood of this intersection we can introduce the common set of the action variables (2.10) for the both systems. So, in this small subvariety of the phase space $\mathcal{M}$ the function $v(p, q) = v(s)$ is a constant of motion.

Now let us consider the known Bäcklund transformation B_ν for the Henon–Heiles system [8, 22], which can be described by the generating function

$$\begin{aligned} F(x, y \mid X, Y) &= -\sqrt{6a\nu}\,yY + \tfrac{2}{5}\sqrt{6a(\nu - x - X)} \\ &\quad \times \left(2\nu^2 + (x + X)\nu + 2x^2 - xX + 2X^2 + \tfrac{5}{4}(y^2 + Y^2) + \tfrac{5}{6}b\right). \end{aligned} \tag{2.11}$$

The Bäcklund transformation B_ν preserves the spectrum of the Lax matrix $\mathcal{L}(\lambda)$ (2.6) (for instance, see [8, 12])

$$\mathsf{M}(\lambda, \nu)\mathcal{L}(\lambda, x, y) = \mathcal{L}(\lambda, X, Y)\mathsf{M}(\lambda, \nu) \tag{2.12}$$

where

$$\mathsf{M}(\lambda, \nu) = \begin{pmatrix} z & 1 \\ 6a(\lambda - \nu) + z^2 & z \end{pmatrix} \qquad z = -\sqrt{6a(\nu - x - X)}. \tag{2.13}$$

Canonical transformation of the extended phase space (2.4) maps the initial Mumford system into the other Mumford system [11] and preserves the spectral curve and entries $\mathbb{A}(\lambda)$, $\mathbb{B}(\lambda)$ of the Lax matrix. Recall, according to the algebro-geometric description of the Bäcklund transformation for the Mumford systems, namely the curve and these entries determine the matrix M. Thus, the mapping (2.4) has to preserve the matrix M. To substitute a new Lax matrix $\widetilde{\mathcal{L}}(\lambda)$ into the Darboux equation (2.12) with the same matrix M (2.13) we have to shift the initial generating function (2.11)

$$\widetilde{F} = F + \frac{z}{6a}\widetilde{H}.$$

Here the Hamiltonian $\widetilde{H}$ is an independent variable of the extended phase space $\mathcal{M}_E$. This means that all the partial derivatives of $\widetilde{H}$ with respect to any other coordinates of $\mathcal{M}_E$ are equal to zero. Thus, the generating function $\widetilde{F}$ gives rise to canonical Bäcklund transformations of the extended phase space $\mathcal{M}_E$.

3. The Toda lattice

The equations of motion of the periodic Toda lattice derive from the following Hamilton function:

$$H(p,q) = \tfrac{1}{2}\sum_{i=1}^{n} p_i^2 + a_i \mathrm{e}^{q_i - q_{i+1}} \qquad a_i \in \mathbb{R}. \tag{3.1}$$

They take the form

$$\dot{q}_i = p_i \qquad \dot{p}_i = a_{i-1}\mathrm{e}^{q_{i-1}-q_i} - a_i \mathrm{e}^{q_i - q_{i+1}}. \tag{3.2}$$

Here $\{p_i, q_i\}$ are canonical variables and the periodicity conventions $q_{i+n} = q_i$ and $p_{i+n} = p_i$ are always assumed for the indices of q_i and p_i.

The exact solution of the equations of motion is due to the existence of the Lax representation [4, 9]

$$\{H(p,q), \mathcal{L}\} = [\mathcal{L}, \mathcal{A}].$$

Here $n \times n$ Lax matrices [4, 9] for the Toda lattice are given by

$$\begin{aligned} \mathcal{L}^{(n)}(\mu) &= \sum_{i=1}^{n} p_i E_{i,i} + \sum_{i=1}^{n-1}\left(\mathrm{e}^{q_i - q_{i+1}} E_{i+1,i} + a_i E_{i,i+1}\right) + \mu \mathrm{e}^{q_n - q_1} E_{1,n} + a_n \mu^{-1} E_{n,1} \\ \mathcal{A}^{(n)}(\mu, q) &= \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} E_{i+1,i} + \mu \mathrm{e}^{q_n - q_1} E_{1,n} \end{aligned} \tag{3.3}$$

where $E_{i,k}$ denotes the $n \times n$ matrix with unity on the intersection of the ith row and the kth column as the only non-zero entry.

According to [19], canonical transformation of the extended phase space (1.1)

$$v(p,q) = v(q) = \exp(q_j - q_{j+1}) \qquad \widetilde{H} = \mathrm{e}^{q_{j+1} - q_j}(H - b) \quad b \in \mathbb{R} \tag{3.4}$$

maps the Toda lattice into the dual integrable system with the following equations of motion:

$$\frac{\mathrm{d}q_i}{\mathrm{d}\widetilde{t}} = v^{-1}(q)\dot{q} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}\widetilde{t}} = v^{-1}(q)\dot{p}_i + \widetilde{H}(\delta_{i,j} - \delta_{i,j+1}). \tag{3.5}$$

Associated with the different indices j, canonical mappings (3.4) are related to each other by canonical transformations of the other variables (p, q).

Mapping (3.4) gives rise to the following transformation of the Lax matrices:

$$\widetilde{L}(\mu) = L(\mu) - \widetilde{H} E_{j,j+1} \qquad \widetilde{A}(\mu) = v^{-1}(q)\, A(\mu). \tag{3.6}$$

At $a_i = 1$ the Poisson brackets relations for the $n \times n$ Lax matrices can be expressed in the r-matrix form

$$\{\overset{1}{\mathcal{L}}(\mu), \overset{2}{\mathcal{L}}(\nu)\} = [r_{12}(\mu, \nu), \overset{1}{\mathcal{L}}(\mu)] + [r_{21}(\mu, \nu), \overset{2}{\mathcal{L}}(\nu)].$$

Here we used the standard notation

$$\overset{1}{\mathcal{L}}(\mu) = \mathcal{L}(\mu) \otimes I \qquad \overset{2}{\mathcal{L}}(\nu) = I \otimes \mathcal{L}(\nu)$$
$$r_{21}(\mu, \nu) = -\Pi r_{12}(\nu, \mu)\Pi$$

and Π is the permutation operator in $\mathbb{C}^n \times \mathbb{C}^n$ [5]. Canonical transformation of the extended phase space (3.4) maps the constant r-matrix for the Toda lattice

$$r_{12}(\mu, \nu) = r_{12}^{const}(\mu, \nu) = \frac{1}{\mu - \nu}\left(\nu \sum_{m \geqslant i} + \mu \sum_{m<i}\right) E_{im} \otimes E_{mi}$$

into the following dynamical r-matrix:

$$\widetilde{r}_{12}(\mu, \nu) = r_{12}^{const}(\mu, \nu) + r_{12}^{dyn}(\mu, \nu) \qquad r_{12}^{dyn}(\mu, \nu) = \widetilde{\mathcal{A}}(\nu, q) \otimes E_{j,j+1}$$

where the second Lax matrix $\widetilde{\mathcal{A}}(\nu, q)$ and, therefore, dynamical r-matrix $\widetilde{r}_{12}$ depend on coordinates only.

The corresponding transformation of the spectral curves looks like

$$\begin{aligned} \mathcal{C}: \quad & -\mu - \frac{\prod_{i=1}^n a_i}{\mu} = P(\lambda) = \lambda^n + \lambda^{n-1}\boldsymbol{p} + \lambda^{n-2}\left(\tfrac{1}{2}\boldsymbol{p}^2 - H\right) + \sum_{i=1}^{n-3} J_i \lambda^i \\ \widetilde{\mathcal{C}}: \quad & -\mu - \frac{(a_j - \widetilde{H})\prod_{i\neq j}^n a_i}{\mu} = \widetilde{P}(\lambda) = \lambda^n + \lambda^{n-1}\boldsymbol{p} + \lambda^{n-2}\left(\tfrac{1}{2}\boldsymbol{p}^2 - b\right) + \sum_{i=1}^{n-3} \widetilde{J}_i \lambda^i. \end{aligned} \tag{3.7}$$

Here $\boldsymbol{p} = J_1 = \sum p_i$ is a total momentum, H and $\widetilde{H}$ are the corresponding Hamilton functions and J_i, $\widetilde{J}_i$ are integrals of motion.

Using the standard form of the hyperelliptic curves C and $\widetilde{C}$ (3.7) and by applying Arnold's method [1, 4], action variables have the form

$$\begin{aligned} s_i &= \oint_{A_i} \frac{1}{2}\left(P(\lambda) + \sqrt{P(\lambda)^2 - 4\prod_{i=1}^n a_i}\right) \mathrm{d}\lambda \\ \widetilde{s}_i &= \oint_{\widetilde{A}_i} \frac{1}{2}\left(\widetilde{P}(\lambda) + \sqrt{\widetilde{P}(\lambda)^2 - 4(a_j - \widetilde{H})\prod_{i\neq j}^n a_i)}\right) \mathrm{d}\lambda \end{aligned} \tag{3.8}$$

where A_i and $\widetilde{A}_i$ are A-cycles of the Jacobi variety of the algebraic curves (3.7), respectively [4]. In fact, polynomials $P(\lambda)$, $\widetilde{P}(\lambda)$ and A-cycles depend on the values of constants of motion, which are dropped in the notation. The Abel transformation linearizes equations of motion by using first kind Abelian differentials on the corresponding spectral curves.

Let parameters a_i determine the potential of the Toda lattice (3.1) and parameters $\widetilde{a}_i$ and b define the potential of the dual system (3.4). At the special choice of the values of integrals of motion

$$H = b \qquad \widetilde{H} = \widetilde{a}_j - \frac{\prod^n a_i}{\prod_{i\neq j}^n \widetilde{a}_i} \qquad J_i = \widetilde{J}_i = \alpha_i \tag{3.9}$$

the initial spectral curve is equal to the resulting curve (3.7). Thus, as for the Maupertuis–Jacobi mapping [2], integral trajectories of the Toda lattice coincide with the trajectories of the dual system on the intersection of the corresponding common levels of integrals $\mathcal{M}_\alpha$ and $\widetilde{\mathcal{M}}_\alpha$. In the neighbourhood of this intersection we can introduce the common set of action variables (3.8) for both systems. So, in this small subvariety of the phase space $\mathcal{M}$ function $v(p,q) = v(s)$ depends on the action variables only.

Another 2×2 Lax representation [7, 15] for the same Toda lattice is equal to

$$T(\lambda) = L_1(\lambda)\cdots L_n(\lambda) \qquad L_i = \begin{pmatrix} \lambda + p_i & \mathrm{e}^{q_i} \\ -a_{i-1}\mathrm{e}^{-q_i} & 0 \end{pmatrix} \tag{3.10}$$

such that

$$\frac{\mathrm{d}L_i}{\mathrm{d}t} = L_i A_i - A_{i-1} L_i \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = [T^{(1\ldots n)}, A_n]$$

where

$$A_i = \begin{pmatrix} \lambda & \mathrm{e}^{q_i} \\ -a_i\mathrm{e}^{-q_{i-1}} & 0 \end{pmatrix}. \tag{3.11}$$

Canonical transformation of the extended phase space (3.4) gives rise to the following transformation of the Lax matrices

$$\begin{aligned} \widetilde{T}(\lambda) &= L_1\cdots L_{j-1}\left[L_j L_{j+1} + \begin{pmatrix} H-b & 0 \\ 0 & 0 \end{pmatrix}\right] L_{j+2}\cdots L_n \\ \widetilde{A}_n(\lambda, q) &= v^{-1}(q)\, A_n(\lambda, q). \end{aligned} \tag{3.12}$$

Note, that for the second Lax matrices A we always have a same form of the transformations (3.6) and (3.12) for all the considered integrable models [17–20].

At $a_i = 1$ the Poisson bracket relations for the 2×2 Lax matrices $T(\lambda)$ (3.10) satisfy the following Sklyanin r-matrix relation:

$$\{\overset{1}{T}(\lambda), \overset{2}{T}(\nu)\} = [R(\lambda-\nu), \overset{1}{T}(u)\overset{2}{T}(\nu)] \qquad R(\lambda-\nu) = \frac{\Pi}{\lambda-\nu}. \tag{3.13}$$

Mapping (3.4) transforms these quadratic relations into the following polylinear ones:

$$\{\overset{1}{\widetilde{T}}(\lambda), \overset{2}{\widetilde{T}}(\nu)\} = [R(\lambda-\nu), \overset{1}{\widetilde{T}}(\lambda)\overset{2}{\widetilde{T}}(\nu)] + [r_{12}^{dyn}(\lambda,\nu), \overset{1}{\widetilde{T}}(\lambda)] + [r_{21}^{dyn}(\lambda,\nu), \overset{2}{\widetilde{T}}(\nu)].$$

The corresponding dynamical r-matrix is given by

$$r_{12}^{dyn}(\lambda,\nu) = A_n(\lambda, q)\otimes\left(L_1\cdots L_{j-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} L_{j+1}\otimes L_n\right).$$

Here all the matrices L_k depend on the spectral parameter ν and $A_n(\lambda, q)$ is the second Lax matrix (3.10).

Now let us look at the separated variables in framework of the traditional consideration of the Toda lattice. Complete list of references can be found in [9, 4, 15]. Below, we put $a_i = 1$ and $j = 1$ without loss of generality, such that

$$\widetilde{H} = \exp(q_2 - q_1)(H+b) \qquad \widetilde{T} = \left[L_1 L_2 + \begin{pmatrix} H+b & 0 \\ 0 & 0 \end{pmatrix}\right] L_3\cdots L_n.$$

This transformation changes the first row of the Lax matrix $T(\lambda)$ only

$$T = \begin{pmatrix} \mathbb{A}(\lambda) & \mathbb{B}(\lambda) \\ \mathbb{C}(\lambda) & \mathbb{D}(\lambda) \end{pmatrix} \quad \mapsto \quad \widetilde{T} = \begin{pmatrix} \widetilde{\mathbb{A}}(\lambda) & \widetilde{\mathbb{B}}(\lambda) \\ \mathbb{C}(\lambda) & \mathbb{D}(\lambda) \end{pmatrix}. \tag{3.14}$$

The separation variables $\{\lambda_1\lambda_2\ldots,\lambda_{n-1}\}$ for the both system are zeros of the non-diagonal common entry

$$\mathbb{C}(\lambda) = \gamma \prod_{i=1}^{n-1}(\lambda - \lambda_i). \tag{3.15}$$

An additional set of variables is defined by the second common entry

$$\mu_i = \mathbb{D}(\lambda_i) \qquad i = 1,\ldots,n-1.$$

Variables $\{\lambda_i, \log\mu_i\}$ are canonically conjugated

$$\{\lambda_i, \log\mu_k\} = \delta_{ik}$$

and the original symplectic form is written as

$$\Omega = \sum_{i=1}^{n-1} \mathrm{d}\log\mu_i \wedge \mathrm{d}\lambda_i + \mathrm{d}\log\gamma \wedge \mathrm{d}\boldsymbol{p}$$

where γ is defined by (3.15). From $\det T(\lambda) = 1$ and $\det \widetilde{T}(\lambda) = (1 - \widetilde{H})$ one immediately obtains one-dimensional equations

$$\begin{aligned} \mathbb{A}(\lambda_i) &= \mu_i^{-1} & \mu_i + \mu_i^{-1} &= P(\lambda_i) \\ \widetilde{\mathbb{A}}(\lambda_i) &= (1 - \widetilde{H})\mu_i^{-1} & \mu_i + (1 - \widetilde{H})\mu_i^{-1} &= \widetilde{P}(\lambda_i). \end{aligned} \tag{3.16}$$

For the special choice of parameters and values of integrals (3.9) the initial separated equations coincide with the resulting ones.

Finally, let us consider the Bäcklund transformation B_ν for the Toda lattice [7, 13]. As is well known [3], transformation B_ν is a canonical transformation $(p, q) \mapsto (P, Q)$ of the initial phase space $\mathcal{M}$ preserving all the integrals of motion

$$I_k(p, q) = I_k(P, Q) \tag{3.17}$$

(see [16] for a more detailed list of properties of B_ν).

For the Toda lattice the canonical transformation B_ν may be described by the following generating function [7]:

$$F_\nu(q \mid Q) = \sum_{i=1}^{n} \left(\mathrm{e}^{q_i - Q_i} - \mathrm{e}^{Q_i - q_{i+1}} - \nu(q_i - Q_i)\right) \tag{3.18}$$

such that

$$p_i = \frac{\partial F}{\partial q_i} \qquad P_i = -\frac{\partial F}{\partial Q_i}. \tag{3.19}$$

To prove that B_ν preserves integrals of motion (3.17) one can verify that B_ν preserves the spectrum of the Lax matrix $L(\mu)$ (3.3)

$$\mathsf{M}(\mu, q, Q)\mathcal{L}(\mu, p, q) = \mathcal{L}(\mu, P, Q)\,\mathsf{M}(\mu, q, Q) \tag{3.20}$$

where

$$\mathsf{M}(\mu, q, Q) = \sum_{i=1}^{n-1} \mathrm{e}^{Q_i - q_{i+1}} E_{i+1,i} + \mu \mathrm{e}^{Q_n - q_1} E_{1,n} \tag{3.21}$$

(see [12] for a detailed account of the theory of the Bäcklund transformation as a gauge transformation).

As integral trajectories of the initial system coincide with trajectories of the resulting system let us substitute a new Lax matrix $\widetilde{\mathcal{L}}(\mu)$ into the Darboux equation (3.20) with the same matrix $\mathsf{M}(\mu, q, Q)$. To resolve the obtained equation for the canonical transformation of the extended phase space at the arbitrary index $1 \leqslant j \leqslant n$ one finds that the new generating function is a shift of the initial generating function

$$\widetilde{F}_\nu(q \mid Q) = F_\nu(q, Q) + \widetilde{H} \mathrm{e}^{Q_j - q_{j+1}} = F_\lambda(q, Q) + \Delta F. \tag{3.22}$$

In (3.22) the Hamiltonian $\widetilde{H}$ has to be considered as an independent variable of the extended phase space $\mathcal{M}_E$. It means that in (3.19) all the partial derivatives of $\widetilde{H}$ with respect to any other coordinates of $\mathcal{M}_E$ are equal to zero. Thus, the new generating function $\widetilde{F}$ gives rise to canonical Bäcklund transformations of the extended phase space $\mathcal{M}_{\widetilde{E}}$.

As above, the same Bäcklund transformations B_ν (3.18) and $\widetilde{B}_\nu$ (3.22) are isospectral deformations of the corresponding 2×2 Lax matrices $T(\lambda)$ and $\widetilde{T}(\lambda)$. For the Toda lattices the intertwining relations are equal to

$$M_i(\lambda, \nu) L_i(p, q) = L_i(P, Q) M_{i+1}(\lambda, \nu)$$

where

$$M_i(\lambda, \nu) = \begin{pmatrix} 1 & \mathrm{e}^{Q_{i-1}} \\ -\mathrm{e}^{-q_i} & \nu - \lambda - \mathrm{e}^{Q_{i-1} - q_i} \end{pmatrix}.$$

The same relations may be used after canonical transformation of the extended phase space at $i \neq j, j+1$. One additional non-factorized relation is given by

$$M_j(\lambda, \nu) \left[L_j(p, q) \begin{pmatrix} \lambda + p_{j+1} & a_j \mathrm{e}^{q_{j+1}} \\ -\mathrm{e}^{-q_{j+1}}(1 + \widetilde{H}) & 0 \end{pmatrix} \right]$$
$$= \left[L_j(P, Q) \begin{pmatrix} \lambda + P_{j+1} & a_j \mathrm{e}^{Q_{j+1}} \\ -\mathrm{e}^{-Q_{j+1}}(1 + \widetilde{H}) & 0 \end{pmatrix} \right] M_{j+2}(\lambda, \nu).$$

The characteristic properties of the new Bäklund transformation $\widetilde{B}_\nu$ are verified following [16]. To prove the spectrality property we have to use one non-factorized relation as well.

Recall, the correspondence between the kernel of the corresponding quantum Baxter $\mathbb{Q}$-operator and the function $F_\lambda(q \mid Q)$ is given by the semiclassical relation [13, 16]. Change of the time (3.4) gives rise to factorization of the $\mathbb{Q}$-operator in the semiclassical limit

$$\widetilde{\mathbb{Q}} \sim \exp(-\mathrm{i}\widetilde{F}/\hbar) = \mathbb{Q} \exp(-\mathrm{i}\Delta F/\hbar).$$

Having obtained a simple change of the separated equations (3.16), one can hope that there is also a simple modification of the one-dimensional Baxter equations in quantum mechanics. Recall, from the Sklyanin work [10, 15] one knows that the eigenfunctions of the quantum Toda lattice Hamiltonian are given by

$$\psi_E(q) = \int C(\lambda, E)\, \psi_\lambda(q)\, \mathrm{d}\lambda \qquad C(\lambda, E) = \prod_{j=1}^{n-1} c(\lambda_j, E).$$

Here ψ_λ are renormalized Whittaker functions and functions $c(\lambda, E)$ satisfy the one-dimensional Baxter equation

$$P(\lambda)c(\lambda, E) = \mathrm{i}^n c(\lambda + \mathrm{i}\hbar, E) + \mathrm{i}^{-n} c(\lambda - \mathrm{i}\hbar, E)$$

where $P(\lambda)$ is a trace of the quantum monodromy matrix $T(\lambda)$. In the classical limit polynomial $P(\lambda)$ enters in the spectral curve (3.7).

Using a similar approach [10, 15], we can suppose that the eigenfunctions for the dual system are expressed in terms of the same Whittaker functions

$$\widetilde{\psi}_{\widetilde{E}}(q) = \int \widetilde{C}(\lambda, \widetilde{E})\psi_\lambda(q)\,\mathrm{d}\lambda$$

whereas the corresponding one-dimensional Baxter equation has to be changed

$$\widetilde{P}(\lambda)\,\widetilde{c}\,(\lambda, \widetilde{E}) = \mathrm{i}^n\left(1 - \widetilde{E}\right)\widetilde{c}\,(\lambda + \mathrm{i}\hbar, \widetilde{E}) + \mathrm{i}^{-n}\,\widetilde{c}\,(\lambda - \mathrm{i}\hbar, \widetilde{E})$$

in accordance with the corresponding classical separated equations (3.16). In the classical limit the polynomial $\widetilde{P}(\lambda)$ enters in the spectral curve (3.7).

It is known in classical mechanics that the harmonic oscillator may be mapped into the Coulomb model by using a canonical change of the time (see references within [19]). In quantum mechanics, the similar duality of the corresponding eigenvalue problems has been used by Schrödinger, Fok and many others. For instance, in the Birman–Schwinger formalism we can estimate the spectrum of the Hamiltonian $\widetilde{H}$ by using the known spectrum of the dual Hamiltonian H. So, it will be interesting to study such a duality within the framework of the quantum $\mathbb{Q}$-operator theory, as an example for the Toda lattice.

4. Conclusion

We discuss canonical transformations of the extended phase space $\mathcal{M}_E$, which are associated with the various parametric forms of a common Lagrangian submanifold. According to the Liouville–Arnold theorem [1], integrability is a geometric property and, therefore, the above mentioned transformations of $\mathcal{M}_E$ preserve the integrability.

So, using the known Lagrangian submanifold of some integrable system we can try to obtain new integrable models related to various parametric forms of this submanifold. In this case, we can expect that the initial and resulting integrable systems have a lot of common properties.

In this paper, starting with the Toda lattice and the Henon–Heiles system we construct another integrable system with the same integral trajectories, separated variables and action variables. The shift of the generating function of the corresponding Bäcklund transformation is given in an obvious form. We can see that the additional term to the generating function is proportional to the Hamilton function for the both integrable systems.

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